Lecture 7

- Principal eigenvalues of differential operators can give the intrinsic growth rates for spatially structured populations described by reaction-diffusion equations.

- Analogue to principal of the Leslie matrix for an age or stage structured population with a finite number of discrete ages or stages (Caswell 1989)

Also

- Metapopulation capacity in Hanski-type metapopulation models (Hanski and Ovaskainen 2000)

- This approach has its origins in the papers of Skellam (1951) and Kierstead and Slobodkin (1953) and the models they studied are frequently called KiSS models (for Kierstead, Slobodkin and Skellam). Here

$u(x,t)$ is the population density on a region Ω where the intrinsic rate of growth of the population is r and the diffusion coefficient is d. The simplest sort of model takes the form

$$\begin{aligned} u_t &= d\Delta u + ru && \text{in } \Omega \times (0,\infty) \\ u &= 0 && \text{on } \partial\Omega \times (0,\infty) \end{aligned} \tag{7.1}$$

(The boundary condition forces the model to be spatially heterogeneous. There is no non-zero spatially constant solution. Were the Dirichlet condition replaced with a Neumann condition $u(x,t) = e^{rt}$ would be a spatially homogeneous solution.)

As in Strauss (2008), solutions to (7.1) can be found via separation of variables in terms of solutions to the related eigenvalue problem

$$\begin{aligned} d\Delta\psi + r\psi &= \sigma\psi && \text{in } \Omega \\ \psi &= 0 && \text{on } \partial\Omega \end{aligned} \tag{7.2}$$

It is a basic fact (Strauss 2008) that (7.2) has an infinite sequence of eigenvalues

$$\sigma_1 > \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_k \geq \cdots$$

with $\sigma_k \to -\infty$ as $k \to \infty$. (Here $\sigma_i > \sigma_{i+1} = \sigma_{i+2} > \sigma_{i+3}$ means σ_{i+1} is an eigenvalue of multiplicity 2. σ_1 is definitely a simple eigenvalue and all eigenvalues have finite multiplicity.)

We will frequently normalize the eigenfunction ψ_k corresponding to σ_k by requiring $\int_\Omega \psi_k^2 = 1$. Notice that if $\sigma_k \neq \sigma_j$,

$$\sigma_k \int_\Omega \psi_k \psi_j = \int_\Omega (d\nabla^2 \psi_k + r\psi_k)\psi_j$$

$$= \int_\Omega (d\nabla^2 \psi_j + r\psi_j)\psi_k \quad \text{by Green' Second Identity}$$

$$= \sigma_j \int_\Omega \psi_j \psi_k$$

$$\Rightarrow \quad (\sigma_k - \sigma_j) \int_\Omega \psi_j \psi_k = 0 \Rightarrow \int_\Omega \psi_j \psi_k = 0$$

We may write solutions to (7.1) as

$$u(x,t) = \sum_{k=1}^{\infty} u_k e^{\sigma_k t} \psi_k(x) \tag{7.3}$$

where the coefficients u_k in (7.3) depend on the initial data $u(x,0)$.

The largest eigenvalue here, σ_1, is referred to as the principal eigenvalue of (7.2). Its associated eigenfunction is always of one sign inside Ω and may be chosen positive. The solution (7.3) will grow exponentially if $\sigma_1 > 0$ but decay exponentially if $\sigma_1 < 0$.

We get a linearized prediction of persistence when $\sigma_1 > 0$

$\Leftrightarrow$ $u \equiv 0$ is an unstable equilibrium of (7.1).

Example $\Omega = (0, \ell)$ in $\mathbb{R}^1$. Then (7.2) becomes

$$\begin{aligned} d\psi'' + r\psi &= \sigma\psi \\ \psi(0) = 0 &= \psi(\ell) \end{aligned} \tag{7.4}$$

Then $\psi'' + \frac{(r-\sigma)}{d}\psi = 0$ in $(0,\ell)$ with $\psi(0) = 0 = \psi(\ell)$

$$\Rightarrow \quad \frac{r-\sigma}{d} = \frac{\pi^2 k^2}{\ell^2} \quad \text{with } \psi_k(x) = \frac{2}{\ell}\sin\left(\frac{\pi k x}{\ell}\right)$$

So then $\sigma_k = r - \frac{d\pi^2 k^2}{\ell^2}$ for $k = 1, \ldots$.

Thus we obtain that

$$\sigma_1 > 0 \iff r - \frac{d\pi^2}{\ell^2} > 0$$

(7.5) $$\iff \frac{r}{d} > \frac{\pi^2}{\ell^2}$$

$$\iff \ell > \sqrt{\frac{d}{r}}\,\pi$$

The inequalities $\frac{r}{d} > \frac{\pi^2}{\ell^2}$ or $r - \frac{d\pi^2}{\ell^2}$ tell how large the local intrinsic growth rate needs to be to offset the loss of individuals through the boundary in order to have a net average intrinsic growth rate over the domain, while the inequality $\ell > \left(\sqrt{\frac{d}{r}}\right)\pi$ in (7.4) indicates the minimum patch size need to support a population in the model. For smaller patches, the population in essence is close enough to the boundary $\partial\Omega$ $(= \{0, \ell\})$ that the loss rate $d\pi^2/\ell^2$ from individuals from dispersal out of Ω is greater than the local population growth rate, r.

If Ω were a square with area A (and thus side $\sqrt{A}$), $\sigma_1 = r - \frac{2d\pi^2}{A}$

$$\psi_1 = \left(\frac{4}{A}\right) \sin\left(\frac{\pi x}{\sqrt{A}}\right) \sin\left(\frac{\pi y}{\sqrt{A}}\right)$$

so that $\sigma_1 > 0 \iff A > \frac{2d\pi^2}{r}$.

The eigenvalues in (7.4) (more generally (7.2)) depend on both the biological parameters d and r

and the size and geometry of Ω. Let's try to separate the biology from the geometry by considering a related eigenvalue problem

(7.6)
$$\Delta \phi + \lambda \phi = 0 \quad \text{in } \Omega$$
$$\phi = 0 \quad \text{on } \partial\Omega$$

(or, equivalently $\quad -\Delta\phi = \lambda\phi \quad$ in Ω
$$\phi = 0 \quad \text{on } \partial\Omega \text{)}$$

Suppose ϕ is an eigenfunction for (7.6).

Let $\psi = \phi$. Then

$$d\Delta\psi + r\psi$$
$$= -d\lambda\psi + r\psi$$
$$= (r - d\lambda)\psi$$

So the eigenvalues of (7.2) are related to those of (7.6) by

(7.7) $$\sigma = r - d\lambda$$

So $\sigma_1 > 0 \iff \frac{r}{d} > \lambda_1$, where λ_1 is the principal eigenvalue of (7.6). Here the eigenvalues of (7.6) are

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k$$

with $\lambda_k \to +\infty$ as $k \to \infty$. If $\Omega = (0, \ell)$, $\lambda_1 = \pi^2/\ell^2$ and if Ω is a square with area A, $\lambda_1 = 2\pi^2/A$. Notice that λ_1 encodes both the geometry of Ω and the hypothesis that the boundary of Ω is absorbing (an assumption about what organisms do when they meet the boundary).

Suppose now that Ω is rescaled to

$\tilde{\Omega} = \{\rho x : x \in \Omega\}$ (If $0 < \rho < 1$, we are shrinking or contracting Ω without altering its shape; if $\rho > 1$,

we are expanding. Suppose ϕ is an eigenfunction for (7.6).

Let $\tilde{\phi}(z) = \phi\left(\frac{z}{\rho}\right)$ for $z \in \tilde{\Omega}$. Then

$$\tilde{\phi}_{z_i}(z) = \phi_{x_i}\left(\frac{z}{\rho}\right)\frac{1}{\rho}$$

$$\tilde{\phi}_{z_i z_i}(z) = \phi_{x_i x_i}\left(\frac{z}{\rho}\right)\frac{1}{\rho^2}$$

$$\Rightarrow \Delta_z \tilde{\phi}(z) = \left(\frac{1}{\rho^2}\right)\Delta_x \phi\left(\frac{z}{\rho}\right)$$

$$= -\frac{\lambda}{\rho^2}\phi\left(\frac{z}{\rho}\right) = -\frac{\lambda}{\rho^2}\tilde{\phi}(z)$$

Thus $\tilde{\phi}$ satisfies

(7.8)
$$\Delta\tilde{\phi} + \frac{\lambda}{\rho^2}\tilde{\phi} = 0 \quad \text{in } \tilde{\Omega}$$
$$\tilde{\phi} = 0 \quad \text{on } \partial\tilde{\Omega}$$

So the eigenvalues $\tilde{\lambda}_k$ for (7.8) satisfy

$$\tilde{\lambda}_k = \lambda_k / \rho^2$$

where λ_k is the corresponding eigenvalue for

Ω in (7.6). So the condition for persistence of a population in the model

$$u_t = d\Delta u + ru \quad \text{in } \tilde{\Omega} \times (0,\infty)$$

$$u = 0 \quad \text{on } \partial\tilde{\Omega} \times (0,\infty)$$

is $r/d > \tilde{\lambda}_1 = \lambda_1/\rho^2$. As ρ becomes larger, λ_1/ρ^2 becomes smaller and a prediction of persistence becomes easier to come by, and analogously, if ρ becomes smaller. Notice that the scaling depends on the order of the elliptic operator, not the dimension of the underlying habitat.

Other Forms of Heterogeneity

There are forms of spatial heterogeneity beyond an absorbing boundary, many leading to models with spatially varying coefficients. It is not usually possible to solve the associated eigenvalue problems explicitly, even in simple geometries.

The classical variational theory of eigenvalues was treated by Courant and Hilbert (1953) and is described in standard texts such as Strauss (2008). Eigenvalues that can be treated variationally typically arise from models of the form

(7.9)
$$u_t = \nabla \cdot d(x) \nabla u + m(x) u \qquad \text{in } \Omega \times (0, \infty)$$

$$d(x) \frac{\partial u}{\partial \eta} + \beta(x) u \qquad \text{on } \partial\Omega \times (0, \infty)$$

<u>or</u>

$$u = 0 \qquad \text{on } \partial\Omega \times (0, \infty)$$

(In general, it is problematic to treat models with advection variationally.) The eigenvalue problem associated with (7.9) is

(7.10)
$$\nabla \cdot d(x) \nabla \psi + m(x) \psi = \sigma \psi \qquad \text{in } \Omega$$

$$\alpha(x)\frac{\partial\psi}{\partial\eta} + \beta(x)\psi = 0 \qquad \text{on } \partial\Omega$$

or

$$\psi = 0 \qquad \text{on } \partial\Omega$$

As with (7.2) the principal eigenvalue in (7.10) can be regarded as the average growth rate over Ω. Moreover, as with (7.2), (7.10) has a decreasing sequence of real eigenvalues, σ_k, with σ_1 simple and having an eigenfunction of one sign on Ω, that tends to $-\infty$ as $k \to \infty$.

Theorem 7.1 (C.C. Thm 2.1) (Courant and Hilbert 1953)

Suppose that $\partial\Omega$ is piecewise of class $C^{2+\alpha}$ and that Ω satisfies an interior cone condition (i.e. there is a cone of fixed size and shape that can be oriented to fit inside Ω if its vertex is at any point on $\partial\Omega$).

Suppose that $d(x) \in C^{1+\alpha}(\bar{\Omega})$ with $d(x) \geq d_0 > 0$;

that $m(x) \in L^\infty(\Omega)$ and $\beta(x) \in L^\infty(\partial\Omega)$ with

$\beta(x) \geq 0$. The principal eigenvalue of (7.10) is

given by:

(a) $\sigma_1 = \max\limits_{\substack{\psi \in W^{1,2}(\Omega) \\ \psi \neq 0}} \left[\dfrac{-\int_\Omega d(x)|\nabla\psi|^2 dx + \int_\Omega m(x)\psi^2 dx - \int_{\partial\Omega} \beta(x)\psi^2 dS}{\int_\Omega \psi^2 dx} \right]$

or (b)

$$\sigma_1 = \max \left\{ -\int_\Omega d(x)|\nabla\psi|^2 dx + \int_\Omega m(x)\psi^2 dx - \int_{\partial\Omega} \beta(x)\psi^2 dS \;\middle|\; \psi \in W^{1,2}(\Omega),\ \int_\Omega \psi^2 dx = 1 \right\}$$

(In the case of Dirichlet boundary conditions on a subset of

$\partial\Omega$, say Γ, $W^{1,2}(\Omega)$ is replaced by the subspace of $W^{1,2}(\Omega)$ which are zero in the appropriate sense on Γ.

This space is $W^{1,2}_0(\Omega)$ if $u = 0$ on $\partial\Omega$.)

Notes: (a) The function ψ_1 corresponding to σ_1 may be chosen so that $\psi_1 > 0$ in Ω; in fact, $\psi_1 > 0$ on $\bar{\Omega}$ except for Dirichlet boundary data.

(b) σ_1 depends continuously on $m(x)$ with respect to $L^p(\Omega)$ for any $p \in (0, \infty]$ if the space dimension $n = 1, 2$ or for $p > n/2$ when $n \geq 3$ (de Figueiredo 1982), continuously on $d(x)$ and $\beta(x)$ on $L^\infty(\Omega)$ and $L^\infty(\partial\Omega)$, respectively.

(c) σ_1 is simple.

(d) Theorem 7.1 allows Ω to have corners but not cusps.

(e) The hypotheses on Ω imply that $\psi \in W^{1,2}(\Omega)$ has a trace in $L^2(\partial\Omega)$. (Adams 1975 Thm 5.22)

(f) The condition $d(x) \in C^{1+\alpha}(\bar{\Omega})$ plus the conditions on $\partial\Omega$ allow an application of elliptic regularity results (Gilbarg and Trudinger 1977; Friedman 1976) $\Rightarrow$ if $\psi \in W^{1,2}(\Omega)$, then in fact $\psi \in W^{2,p}(\Omega)$ for any $p \Rightarrow$ $\psi \in C^{1+\alpha}(\bar{\Omega}) \Rightarrow$ boundary condition makes sense in classical terms.

(g) It may be necessary to apply the regularity results in an iterative fashion to obtain the desired degree of smoothness.

Corollary 7.2 (CC Corollary 2.2) Let $\sigma_1(d, m, \beta)$ denote the principal eigenvalue of (7.10). If $m_1 \geq m_2$, then $\sigma_1(d, m_1, \beta) \geq \sigma_1(d, m_2, \beta)$. If $m_1 > m_2$ on a subset of Ω of positive measure, $\sigma_1(d, m_1, \beta) > \sigma(d, m_2, \beta)$. Similarly, $\sigma_1(d, m, \beta)$ is decreasing with respect to β in the same sense and also with respect to d unless $\beta \equiv 0$ and $m(x)$ is constant. (In this last case, ψ_1 is constant and the function $d(x)$ has no effect on σ_1.)

Proof: Let ψ_1 be the eigenfunction associated with $\sigma_1(d, m_2, \beta)$. ψ_1 is the maximizer for the quotient in the statement of Theorem 7.1, corresponding to $\sigma_1(d, m_2, \beta)$.

So

$$\sigma_1(d, m_2, \beta) = \frac{-\int_\Omega d|\nabla\psi_1|^2 + \int_\Omega m_2 \psi_1^2\,dx - \int_{\partial\Omega} \beta\psi_1^2\,ds}{\int_\Omega \psi_1^2\,dx}$$

$$\leq \frac{-\int_\Omega d|\nabla\psi_1|^2\,dx + \int_\Omega m_1(x)\psi_1^2\,dx - \int_{\partial\Omega} \beta\psi_1^2\,ds}{\int_\Omega \psi_1^2\,dx}$$

$$\leq \max_{\substack{\psi \in W^{1,2}(\Omega) \\ \psi \neq 0}} \frac{-\int_\Omega d|\nabla\psi|^2 dx + \int_\Omega m_1(x)\psi^2 dx - \int_{\partial\Omega} \beta\psi^2 dS}{\int_\Omega \psi^2 dx}.$$

$$= \sigma_1(d, m_1, \beta)$$

Since $\psi_1 > 0$ in Ω, the first inequality is strict so long as $m_1(x) > m_2(x)$ on a set of positive measure.

Notes: (i) $m(x)$ denotes the local per capita growth rate and σ_1 measures the average growth rate over Ω. So Corollary 7.2 is clearly natural wrt m.

(ii) β describes how likely it is for an individual reaching $\partial\Omega$ to leave Ω. If $d(x) \equiv 1$, $\beta(x) = \beta_0$ the boundary condition in (7.10) can be expressed as

$$(1-p)\frac{\partial u}{\partial \eta} + pu = 0$$

where $p = \beta_0/\beta_0+1$. In such case, p is the fraction of individuals reaching $\partial\Omega$ that leave Ω. In general, increasing β has the effect of increasing the rate at which individuals are lost by dispersal out of Ω.

(iii) Increasing d increases the movement rate within Ω, which increases contact with $\partial\Omega$, and hence the rate of loss through $\partial\Omega$ if $\beta \neq 0$. So Corollary 7.2 is natural wrt d or β.

For the model $u_t = d\Delta u + ru$ in $\Omega \times (0,\infty)$ with $u = 0$ on $\partial\Omega \times (0,\infty)$, the principal eigenvalue is $\sigma_1 = r - d\lambda_1$, where λ_1 is the principal eigenvalue of $\Delta\phi = -\lambda\phi$ in Ω, $\phi = 0$ on $\partial\Omega$. Hence $-\lambda_1$ is the principal eigenvalue σ for the problem

$$\Delta\psi = \sigma\psi \quad \text{in } \Omega$$

$$\psi = 0 \quad \text{on } \partial\Omega$$

So Theorem 7.1 $\Rightarrow$

$$\lambda_1 = -\max_{\substack{\psi \in W_0^{1,2}(\Omega) \\ \psi \neq 0}} \left(\frac{-\int_\Omega |\nabla \psi|^2 dx}{\int_\Omega \psi^2 dx} \right)$$

$$= \min_{\substack{\psi \in W_0^{1,2}(\Omega) \\ \psi \neq 0}} \left(\frac{\int_\Omega |\nabla \psi|^2 dx}{\int_\Omega \psi^2 dx} \right)$$

Corollary 7.3 (CC Corollary 2.3). Let $\lambda_1(\Omega)$ denote the principal eigenvalue of $-\Delta\phi = \lambda\phi$ in Ω, $\phi = 0$ on $\partial\Omega$. Then if $\Omega_1 \subseteq \Omega_2$, $\lambda_1(\Omega_1) \geq \lambda_1(\Omega_2)$, with strictly inequality if $\Omega_2 \setminus \Omega_1$ contains an open set.

Proof: If $\psi \in W_0^{1,2}(\Omega_1)$, we may extend ψ to be zero on $\Omega_2 \setminus \Omega_1$, and the resulting function lies in $W_0^{1,2}(\Omega_2)$. Let ψ_1 be the eigenfunction associated with $\lambda_1(\Omega_1)$. Let $\tilde{\psi}_1$ be its extension to Ω_2 by taking $\tilde{\psi}_1 = 0$ on $\Omega_2 \setminus \Omega_1$. $\tilde{\psi}_1 \in W_0^{1,2}(\Omega_2)$. So $\lambda_1(\Omega_2) =$

$$\min_{\substack{\psi \in W_0^{1,2}(\Omega) \\ \psi \neq 0}} \frac{\int_{\Omega_2} |\nabla\psi|^2}{\int_{\Omega_2} \psi^2} \leq \int_\Omega |\nabla\hat{\psi}_1|^2 dx \Big/ \int_\Omega \hat{\psi}_1^2 dx$$

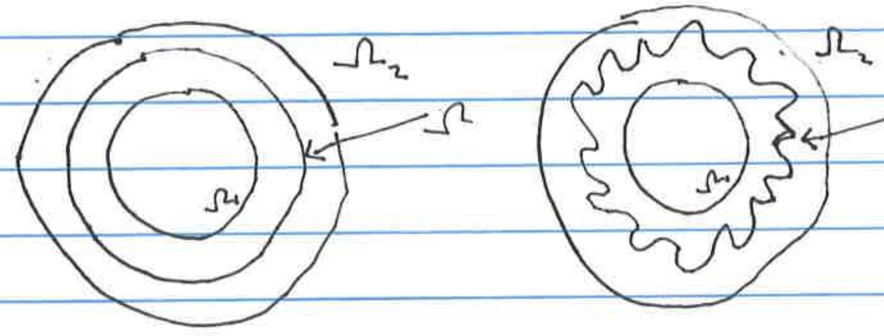
$$= \int_{\Omega_1} |\nabla\psi_1|^2 dx \Big/ \int_\Omega \psi_1^2 dx = \lambda_1(\Omega_1)$$

To obtain the strict inequality, note that the eigenfunction for $\lambda_1(\Omega_2)$ is positive throughout Ω_2, which $\tilde{\psi}_1$ is not. So $\tilde{\psi}_1$ can not be the minimizer of the quotient for $\lambda_1(\Omega_2)$.

What aspects of Ω does $\lambda_1(\Omega)$ measure?

(i) Suppose $\Omega_1 \subseteq \Omega \subseteq \Omega_2$ and $\Omega_1 \subseteq \tilde{\Omega} \subseteq \Omega_2$.

Then $\lambda_1(\Omega), \lambda_1(\tilde{\Omega}) \subseteq (\lambda_1(\Omega_2), \lambda_1(\Omega_1))$



$\lambda_1(\Omega), \lambda_1(\tilde{\Omega})$ close even though the perimeter to area ratio is much higher for $\tilde{\Omega}$ than it is for Ω. (Here we are in 2D)

(ii) Suppose the maximum width of Ω is less than a, so that one is never more than a units from the boundary. By rotating Ω, we can assume Ω lies inside a rectangle R of width a and height b for some $b > 0$. $\lambda_1(R) =$

$$\pi^2\left(\frac{1}{a^2} + \frac{1}{b^2}\right) \geq \frac{\pi^2}{a^2}. \quad \text{So } \lim_{a \to 0} \lambda_1(\Omega) = \infty,$$

even if we selected Ω so that $|\Omega| =$ area of Ω is fixed. (Think of rectangles that become skinnier but longer.)

(iii) $\lambda_1(\Omega)$ essentially measures the core area of Ω, taking into account its size and average distance from $\partial\Omega$. On the other hand, $\lambda_1(\Omega)$ is relative insensitive to changes that increase the length of $\partial\Omega$ but do not affect the core area.

(iv) Noss et al (1989)
Lovejoy et al (1986)
Groom and Schumaker (1990)
Stamps et al (1987)
McKelvey et al (1986)

notion of "core area"

(v) Other methods of estimating eigenvalues

Weinberger (1974)
Schaefer (1988)
Sperb (1981)
Belgacem (1997)

Bandle (1980) : isoperimetric inequality :

Among all planar regions Ω of a given area, $\lambda_1(\Omega)$ is smallest when Ω is a circular disc.

This reflects that a disc would give us the most core area.

When coefficients are variable, the "geometric" eigenvalue problem corresponding to (7.10) becomes

(7.11)
$$\begin{cases} \nabla \cdot d(x) \nabla \phi + \lambda m(x) \phi = 0 & \text{in } \Omega \\ d(x) \dfrac{\partial \phi}{\partial \eta} + \beta(x) \phi = 0 & \text{on } \partial\Omega \end{cases}$$

<u>Theorem</u> 7.4 (CC Thm 2.4) Suppose that Ω and the coefficients d, m and β satisfy the hypotheses of Theorem 7.1. Assume further that $\partial\Omega$ is of class C^1 and that $m(x)$ is positive on an open subset of Ω and $\beta(x)$ is positive on a (relatively) open subset of $\partial\Omega$. Then (7.11) admits a positive principal eigenvalue λ_1^+ determined by

(7.12a)
$$\frac{1}{\lambda_1^+} = \max_{\substack{\phi \in W^{1,2}(\Omega) \\ \phi \neq 0}} \left[\frac{\int_\Omega m \phi^2 dx}{\int_\Omega d |\nabla \phi|^2 + \int_{\partial\Omega} \beta(x) \phi^2 ds} \right]$$

In the case of Dirichlet boundary data

(7.12b) $$\frac{1}{\lambda_1^+} = \max_{\substack{\phi \in W_0^{1,2}(\Omega) \\ \phi \neq 0}} \left[\frac{\int_\Omega m\phi^2 \, dx}{\int_\Omega d\,|\nabla \phi|^2 \, dx} \right]$$

The principal eigenvalue is the only positive eigenvalue admitting a positive eigenfunction and it is a simple eigenvalue. The principal eigenvalue depends continuously on $m(x)$ with respect to $L^p(\Omega)$ for any $p \in (1, \infty]$ in the case of $n = 1, 2$ and for $p > n/2$ when $n \geq 3$.

An outline of the proof is included in the notes and posted online.

The result in the Dirichlet case is originally due to Manes and Micheletti (1973). When $m(x)$ is variable but positive throughout $\bar{\Omega}$, results on principal eigenvalues are in Protter and Weinberger (1966).

To obtain the existence of a principal eigenvalue in Theorem 2.4 we work in the Hilbert space $W^{1,2}(\Omega)$ (or $W_0^{1,2}(\Omega)$ in the case of Dirichlet boundary conditions.) The essential abstract result we shall use is the following. (See de Figueiredo (1982).)

Lemma 2.A.1 Let H be a Hilbert space. Denote the inner product on H by $\langle\cdot,\cdot\rangle$ and the corresponding norm by $||\ \ ||$. Let T be a compact symmetric operator on H. If

$$\mu_1 = \sup\{\langle Tu, u\rangle : ||u|| = 1\} > 0 \tag{2.A.1}$$

then μ_1 is an eigenvalue of T, i.e. there exists $\phi_1 \in H$, $\phi_1 \neq 0$ so that $T\phi_1 = \mu_1\phi_1$.

Proof: If T is compact it must be bounded, so $\mu_1 \leq ||T|| < \infty$. Choose a sequence $\{u_n\}$ with $||u_n|| = 1$ and $\lim\limits_{n\to\infty}\langle Tu_n, u_n\rangle = \mu_1$. A bounded sequence in a Hilbert space must have a weakly convergent subsequence, and a compact operator on a Hilbert space maps weakly convergent sequences into strongly convergent sequences, so we may pass to a subsequence and obtain $u_n \to \phi_1$ weakly and $Tu_n \to T\phi_1$ strongly for some $\phi_1 \in H$. Then $\mu_1 = \lim\limits_{n\to\infty}\langle Tu_n, u_n\rangle = \langle T\phi_1, \phi_1\rangle$. Since $\mu_1 \neq 0$ we must have $\phi_1 \neq 0$ so we may assume that $||\phi_1|| = 1$.) If $w \in H$, $w \neq 0$ then

$$\begin{aligned}\langle (T-\mu_1)w, w\rangle &= \langle Tw, w\rangle - \mu_1||w||^2\\ &= ||w||^2[\langle T(w/||w||), (w/||w||)\rangle - \mu_1]\\ &\leq 0.\end{aligned}$$

(The inequality clearly holds for $w = 0$ as well.) For any fixed $v \in H$, let $w = \phi_1 - tv$. We have for all $t \in I\!R$

$$0 \leq \langle(\mu_1 - T)(\phi_1 - tv), (\phi_1 - tv)\rangle = \langle(\mu_1 - T)\phi_1, \phi_1\rangle - 2t\langle(\mu_1 - T)\phi_1, v\rangle$$
$$+t^2\langle(\mu_1 - T)v, v\rangle,$$

but $\langle(T-\mu_1)\phi_1, \phi_1\rangle = 0$ so we have $2t(\langle(\mu_1 - T)\phi_1, v\rangle) \leq t^2(\langle(\mu_1 - T)v, v\rangle)$. The last inequality cannot be valid for both positive and negative values of t near $t = 0$ unless $\langle(T-\mu_1)\phi_1, v\rangle = 0$. Since $v \in H$ was arbitrary, we must have $\langle(T-\mu_1)\phi_1, v\rangle = 0$ for all $v \in H$, so $(T-\mu_1)\phi_1 = 0$, i.e. $T\phi_1 = \mu_1\phi_1$.

We want to work in the Hilbert space $W^{1,2}(\Omega)$, but we want to replace the standard inner product

$$\langle u, v\rangle_{1,2} = \int_\Omega (\nabla u \cdot \nabla v + uv)dx$$

with the inner product

$$\langle u, v\rangle = \int_\Omega d(x)\nabla u \cdot \nabla v dx + \int_{\partial\Omega} \beta(x)uv dS. \tag{2.A.2}$$

The following is adapted from Mikhailov (1978, Ch. III, §5).

Lemma 2.A.2. Suppose that Ω is a bounded domain with $\partial\Omega$ of class C^1, $d(x)$ is a bounded measurable function with $d(x) \geq d_0$ almost everywhere on Ω, and $\beta(x)$ is a bounded measurable function on $\partial\Omega$ with $\beta(x) \geq 0$ almost everywhere and $\beta(x) > 0$ almost everywhere on a subset of $\partial\Omega$ that is open relative to $\partial\Omega$. Then the inner products $\langle\cdot,\cdot\rangle_{1,2}$ and $\langle\cdot,\cdot\rangle$ generate equivalent norms.

Proof: Since $d(x)$ is bounded it is clear that for some constant C,

$$\int_\Omega d(x)|\nabla u|^2 dx \leq C \int_\Omega (|\nabla u|^2 + u^2)dx.$$

The regularity of $\partial\Omega$ implies that any function $u \in W^{1,2}(\Omega)$ has a trace $tr(u)$ on $\partial\Omega$ which belongs to $L^2(\partial\Omega)$ with $||tr(u)||_{L^2(\partial\Omega)} \leq C||u||_{1,2}$ for some constant C, and the trace operator $tr : W^{1,2}(\Omega) \to L^2(\partial\Omega)$ is compact; see Adams (1975, Theorem 5.22) and Mikhailov (1978, Ch. III). It follows from these relations and the boundedness of $\beta(x)$ that $\langle u, u\rangle \leq C\langle u, u\rangle_{1,2}$.

Suppose that there does not exist any constant C_1 with $\langle u, u\rangle_{1,2} \leq C_1\langle u, u\rangle$. We may then choose a sequence $\{u_n\}$ with $\langle u_n, u_n\rangle = 1$ and $\langle u_n, u_n\rangle_{1,2} = n$; letting $v_n = (1/\sqrt{n})u_n$ we have $\langle v_n, v_n\rangle_{1,2} = 1$ but $\langle v_n, v_n\rangle = 1/n$. Since $\{v_n\}$ is bounded in $W^{1,2}(\Omega)$, the compact embedding of $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ implies that there is a subsequence which converges in $L^2(\Omega)$. Passing to the subsequence we have

$$\begin{aligned}
||v_n - v_m||_{1,2}^2 &= ||v_n - v_m||_{L^2(\Omega)}^2 + ||\nabla(v_n - v_m)||_{L^2(\Omega)}^2 \\
&\leq ||v_n - v_m||_{L^2(\Omega)}^2 + \int_\Omega |\nabla(v_n - v_m)|^2 dx \\
&\leq ||v_n - v_m||_{L^2(\Omega)}^2 + (2/d_0)\int_\Omega d(x)(|\nabla v_n|^2 + |\nabla v_m|^2)dx \\
&\leq ||v_n - v_m||_{L^2(\Omega)}^2 + (2/d_0)(\langle v_n, v_n\rangle + \langle v_m, v_m\rangle) \\
&\leq ||v_n - v_m||_{L^2(\Omega)}^2 + (2/d_0)[(1/n) + (1/m)],
\end{aligned}$$

so that the subsequence is Cauchy and hence convergent in $W^{1,2}(\Omega)$. It follows that $v_n \to v$ in $W^{1,2}(\Omega)$ with $\langle v, v\rangle_{1,2} = 1$. We also have

$$0 \leq \int_\Omega d(x)|\nabla v_n|^2 dx \leq \langle v_n, v_n\rangle = 1/n,$$

$$0 \leq \int_{\partial\Omega} \beta(x)v_n^2 dS \leq \langle v_n, v_n\rangle = 1/n,$$

and $tr(v_n) \to tr(v)$ on $\partial\Omega$; thus

$$\int_\Omega d(x)|\nabla v|^2 dx = 0$$

and

$$\int_{\partial\Omega} \beta(x)v^2 dS = 0.$$

Since $d(x) \geq d_0$ almost everywhere, we must have $\nabla v = 0$ almost everywhere, so v must be a constant. Since $\beta(x) > 0$ on an open subset of $\partial\Omega$, the constant must be zero. However, $v \equiv 0$ contradicts $\langle v, v\rangle_{1,2} = 1$. Thus, our assumption that $\langle u, u\rangle_{1,2}$ is not bounded by $C_1\langle u, u\rangle$ for any C_1 leads to a contradiction, so there must be a C_1 such that $\langle u, u\rangle_{1,2} \le C_1\langle u, u\rangle$.

Proof of Theorem 2.4: For any $u \in W^{1,2}(\Omega)$ we can define a functional $f(u) : v \mapsto \int_\Omega m(x)uv\,dx$. Since $m(x) \in L^\infty(\Omega)$ the functional $f(u)$ is bounded on $W^{1,2}(\Omega)$, because the Schwartz inequality implies $\|f(u)v\| \le \|m\|_\infty\|u\|_2\|v\|_2$ (where $\|\cdot\|_\infty$ and $\|\cdot\|_2$ denote the norms of L^∞ and L^2, respectively), and the norm on $W^{1,2}(\Omega)$ dominates the norm on $L^2(\Omega)$. Thus, by the Riesz representation theorem, there is an element Tu such that $\langle Tu, v\rangle = f(u)v$. Since $f(u)v = f(v)u$ the operator $T : W^{1,2}(\Omega) \to W^{1,2}(\Omega)$ is symmetric, and since $f(u)$ is bounded as a functional with norm $\|f(u)\| \le \|m\|_\infty\|u\|_2 \le c\|m\|_\infty\|u\|$ (where $\|\ \|$ is the norm defined by the inner product in (2.A.2)) the operator T is also bounded. Suppose that $\{u_n\}$ is a bounded sequence in $W^{1,2}(\Omega)$. Since $W^{1,2}(\Omega)$ is a Hilbert space, $\{u_n\}$ has a weakly convergent subsequence. Passing to the subsequence and reindexing, we have $u_n \rightharpoonup u$ (i.e. $\{u_n\}$ converges weakly to u) for some $u \in W^{1,2}(\Omega)$. The space $W^{1,2}(\Omega)$ embeds compactly in $L^2(\Omega)$ (see Adams (1975, Theorem 6.2); see also Gilbarg and Trudinger (1977)) so we have $u_n \to u$ in $L^2(\Omega)$. However,

$$\begin{aligned}
\|Tu_n - Tu\|^2 &= \langle Tu_n - Tu, Tu_n - Tu\rangle \\
&= \int_\Omega m(u_n - u)(Tu_n - Tu)dx \\
&\le \|m\|_\infty\|u_n - u\|_2\|Tu_n - Tu\|_2 \\
&\le c\|m\|_\infty\|u_n - u\|_2\|Tu_n - Tu\|.
\end{aligned}$$

Thus, $\|Tu_n - Tu\| \le c\|m\|_\infty\|u_n - u\|_2$ so $Tu_n \to Tu$ in $W^{1,2}(\Omega)$. This establishes that T is compact. Since $m(x) > 0$ on an open subset of Ω, we can take u to be a function which is nonzero on a set of positive measure inside that subset and zero elsewhere, so that $\langle Tu, u\rangle = \int_\Omega mu^2dx > 0$. It then follows from Lemma 2.A.1 that there exist a principal eigenvalue $\mu_1 > 0$ and corresponding eigenfunction ϕ_1 for T such that $T\phi_1 = \mu_1\phi_1$. Thus, for any $v \in W^{1,2}(\Omega)$, $\langle T\phi_1, v\rangle = \mu_1\langle\phi_1, v\rangle$ or alternatively

$$(1/\mu_1)\int_\Omega m\phi_1 v\,dx = \int_\Omega d(x)\nabla\phi_1\cdot\nabla v\,dx + \int_{\partial\Omega}\beta(x)\phi_1 v\,dS. \tag{2.A.3}$$

Since relation (2.A.3) holds for any $v \in W^{1,2}(\Omega)$, ϕ_1 is a weak solution to

$$\begin{aligned}
\nabla\cdot d(x)\nabla\phi_1 &= (1/\mu_1)m(x)\phi_1 \quad \text{in} \quad \Omega \\
d(x)\frac{\partial\phi_1}{\partial\vec{n}} + \beta(x)\phi_1 &= 0 \qquad \text{on} \quad \partial\Omega.
\end{aligned}$$

By standard elliptic regularity theory, ϕ_1 will belong to $W^{2,p}(\Omega)$ for any p. (If $m(x) \in C^\alpha(\overline{\Omega})$ then $\phi_1 \in C^{2+\alpha}(\overline{\Omega})$.) (See Gilbarg and Trudinger (1977) or Section 1.6.) The positivity of the eigenfunction ϕ follows as in the derivation of Theorem 2.1 in Courant and Hilbert (1953).

Remarks: The space $W^{1,2}(\Omega)$ embeds compactly in L^{p^*} for any $p^* < \infty$ if the dimension of the underlying domain Ω is 1 or 2 and for $p^* < 2N/(N-2)$ if $N \ge 3$. Thus, for such

p we have by repeated applications of Hölder's inequality that

$$|\langle Tu, v\rangle| = |\int_\Omega muv\,dx| \le ||m||_{L^r(\Omega)}||u||_{L^p(\Omega)}||v||_{L^q(\Omega)}$$

if $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$. Taking $p=q=p^*>2$ we have $\langle Tu, v\rangle \le C||m||_{L^r(\Omega)}||u||||v||$ with $r = p^*/(p^*-2)$. The point is that the quadratic form $\langle Tu, v\rangle$ is bounded on $W^{1,2}(\Omega)\times W^{1,2}(\Omega)$ not only in terms of $||m||_\infty$ but in terms of $||m||_{L^r(\Omega)}$ for $r > p^*/(p^*-2)$. In the case of $N=1,2$ any $r>1$ is admissible. If $N=3$ we must have $r>N/2$ because of the restriction on p^*. Thus, we could replace the condition $m\in L^\infty(\Omega)$ with $m \in L^r(\Omega)$, and also we can obtain the continuity of $\langle Tu, v\rangle$ and hence μ_1 with respect to m relative to the norm in L^r. This is done in detail in deFigueiredo (1982).

The formula (2.A.1) as applied in the proof of Theorem 2.4 gives

$$\mu_1 = \frac{1}{\lambda_1^+(m)} = \sup\left\{\int_\Omega mu^2dx : \int_\Omega d|\nabla u|^2dx + \int_{\partial\Omega}\beta u^2 dS = 1\right\}. \tag{2.A.4}$$

If $w\in W^{1,2}(\Omega)$ with $w\neq 0$, use the norm defined by (2.A.2) and let $\tilde{w} = w/||w||$. We have

$$1 = ||\tilde{w}||^2 = \int_\Omega d|\nabla\tilde{w}|^2dx + \int_{\partial\Omega}\beta(x)\tilde{w}^2dS.$$

Also,

$$\int_\Omega m\tilde{w}^2dx = \frac{1}{||w||^2}\int_\Omega mw^2dx = \frac{\int_\Omega mw^2dx}{\int_\Omega d|\nabla w|^2dx + \int_{\partial\Omega}\beta w^2dS}.$$

Thus, taking

$$\sup\left\{\frac{\int_\Omega mw^2dx}{\int_\Omega d|\nabla w|^2dx + \int_{\partial\Omega}\beta w^2dS} : w\in W^{1,2}(\Omega), w\neq 0\right\} \tag{2.A.5}$$

gives the same value $\mu_1 = 1/\lambda_1^+(m)$ as in (2.A.4).

The case of Neumann boundary conditions, as in Theorem 2.5, requires special treatment because $\left(\int d|\nabla u|^2dx\right)^{1/2}$ is not equivalent to the standard norm on $W^{1,2}(\Omega)$ since it is equal to zero for u constant but nonzero. To prove Theorem 2.5 we shall need an auxiliary result which is roughly equivalent to a lemma introduced by Fleming (1975), and is discussed in the context of deriving Theorem 2.5 by Brown and Lin (1980).

Lemma 2.A.3. Suppose that $\int_\Omega m(x)dx < 0$. Let $\{\phi_n\}\in W^{1,2}(\Omega)$ be a sequence with $\int_\Omega \phi_n^2dx = 1$ and $\int_\Omega m\phi_n^2dx > 0$. There is a constant $c_0>0$ such that $\int_\Omega d(x)|\nabla\phi_n|^2dx \ge c_0$ for all n.

Theorem 7.5 (CC Thm 2.5) (Brown and Lin 1980). In the case of Neumann boundary conditions ($\beta \equiv 0$), (7.11) admits a principal eigenvalue $\Leftrightarrow$

(7.13) $$\int_\Omega m(x)\,dx < 0$$

In that case λ_1^+ is given by

(7.14) $$\frac{1}{\lambda_1^+} = \max_{\substack{\phi \in W^{1,2}(\Omega) \\ \phi \neq 0}} \left[\frac{\int_\Omega m\phi^2\,dx}{\int_\Omega d|\nabla\phi|^2\,dx} \right]$$

Notes: (i) Suppose $\int_\Omega m(x) \geq 0$. Then choosing $\psi \equiv 1$ as test function in Theorem 7.1 $\Rightarrow$

$\sigma_1 \geq \int m(x)\,dx / |\Omega| \geq 0$. If $m(x)$ is non-constant, $\psi \equiv 1$ can not be the eigenfunction, so $\sigma_1 > 0$.

(ii) λ_1^+ is increasing in d and β and decreasing in m.

(iii) A sketch of the proof is included in the notes.

p we have by repeated applications of Hölder's inequality that

$$|\langle Tu, v\rangle| = |\int_\Omega muv\, dx| \le ||m||_{L^r(\Omega)}||u||_{L^p(\Omega)}||v||_{L^q(\Omega)}$$

if $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$. Taking $p=q=p^*>2$ we have $\langle Tu, v\rangle \le C||m||_{L^r(\Omega)}||u||||v||$ with $r = p^*/(p^*-2)$. The point is that the quadratic form $\langle Tu, v\rangle$ is bounded on $W^{1,2}(\Omega)\times W^{1,2}(\Omega)$ not only in terms of $||m||_\infty$ but in terms of $||m||_{L^r(\Omega)}$ for $r > p^*/(p^*-2)$. In the case of $N=1,2$ any $r>1$ is admissible. If $N=3$ we must have $r>N/2$ because of the restriction on p^*. Thus, we could replace the condition $m\in L^\infty(\Omega)$ with $m\in L^r(\Omega)$, and also we can obtain the continuity of $\langle Tu, v\rangle$ and hence μ_1 with respect to m relative to the norm in L^r. This is done in detail in deFigueiredo (1982).

The formula (2.A.1) as applied in the proof of Theorem 2.4 gives

$$\mu_1 = \frac{1}{\lambda_1^+(m)} = \sup\left\{\int_\Omega mu^2dx : \int_\Omega d|\nabla u|^2dx + \int_{\partial\Omega}\beta u^2dS = 1\right\}. \tag{2.A.4}$$

If $w\in W^{1,2}(\Omega)$ with $w\ne 0$, use the norm defined by (2.A.2) and let $\tilde{w} = w/||w||$. We have

$$1 = ||\tilde{w}||^2 = \int_\Omega d|\nabla\tilde{w}|^2dx + \int_{\partial\Omega}\beta(x)\tilde{w}^2dS.$$

Also,

$$\int_\Omega m\tilde{w}^2dx = \frac{1}{||w||^2}\int_\Omega mw^2dx = \frac{\int_\Omega mw^2dx}{\int_\Omega d|\nabla w|^2dx + \int_{\partial\Omega}\beta w^2dS}.$$

Thus, taking

$$\sup\left\{\frac{\int_\Omega mw^2dx}{\int_\Omega d|\nabla w|^2dx + \int_{\partial\Omega}\beta w^2dS} : w\in W^{1,2}(\Omega),\ w\ne 0\right\} \tag{2.A.5}$$

gives the same value $\mu_1 = 1/\lambda_1^+(m)$ as in (2.A.4).

The case of Neumann boundary conditions, as in Theorem 2.5, requires special treatment because $\left(\int d|\nabla u|^2dx\right)^{1/2}$ is not equivalent to the standard norm on $W^{1,2}(\Omega)$ since it is equal to zero for u constant but nonzero. To prove Theorem 2.5 we shall need an auxiliary result which is roughly equivalent to a lemma introduced by Fleming (1975), and is discussed in the context of deriving Theorem 2.5 by Brown and Lin (1980).

Lemma 2.A.3. Suppose that $\int_\Omega m(x)dx < 0$. Let $\{\phi_n\}\in W^{1,2}(\Omega)$ be a sequence with $\int_\Omega \phi_n^2dx = 1$ and $\int_\Omega m\phi_n^2dx > 0$. There is a constant $c_0>0$ such that $\int_\Omega d(x)|\nabla\phi_n|^2dx \ge c_0$ for all n.

Proof: Suppose not. Then by passing to a subsequence we may choose ϕ_n so that $\int_\Omega d(x)|\nabla\phi_n|^2 dx \le 1/n$. Since $\int_\Omega \phi_n^2 dx = 1$, the subsequence is bounded in $W^{1,2}(\Omega)$. Since $W^{1,2}(\Omega)$ embeds compactly in $L^2(\Omega)$, there is a subsequence which converges in $L^2(\Omega)$. Passing to that subsequence we have $\phi_n \to \phi$ in $L^2(\Omega)$ so that $\int_\Omega \phi^2 dx = \lim_{n\to\infty} \int_\Omega \phi_n^2 dx = 1$. Also,

$$\begin{aligned}
||\phi_n - \phi_m||_{1,2}^2 &= \int_\Omega |\phi_n - \phi_m|^2 dx + \int_\Omega |\nabla\phi_n - \nabla\phi_m|^2 dx \\
&\le \int_\Omega |\phi_n - \phi_m|^2 dx + (2/d_0)\int_\Omega (d|\nabla\phi_n|^2 + d|\nabla\phi_m|^2) dx \\
&\le \int_\Omega |\phi_n - \phi_m|^2 dx + \frac{2}{d_0}\left(\frac{1}{n} + \frac{1}{m}\right),
\end{aligned}$$

so the sequence is Cauchy and hence convergent in $W^{1,2}(\Omega)$. We have

$$\int_\Omega d|\nabla\phi|^2 dx = \lim_{n\to\infty} \int_\Omega d|\nabla\phi_n|^2 dx = 0.$$

Thus, ϕ must be constant and since $\int_\Omega \phi^2 dx = 1$ we must have $\phi = 1/|\Omega|^{1/2} \neq 0$ so

$$\int_\Omega m\phi^2 dx = 1/|\Omega| \int_\Omega m dx < 0.$$

However, since $m(x)$ is a bounded measurable function, the mapping $\phi_n \mapsto \int_\Omega m\phi_n^2 dx$ is continuous as a mapping from $L^2(\Omega)$ into $I\!R$, so $\int_\Omega m\phi^2 dx = \lim_{n\to\infty}\int_\Omega m\phi_n^2 dx \ge 0$. This yields a contradiction. Thus, there must exist a constant $c_0 > 0$ as asserted by the lemma.

Proof of Theorem 2.5: Our approach follows that of Brown and Lin (1980). Observe that λ_1^+ is a positive principal eigenvalue for

$$\begin{aligned}
\nabla \cdot d(x)\nabla\phi + \lambda m(x)\phi = 0 \text{ on } \quad & \Omega \\
\frac{\partial\phi}{\partial\vec{n}} = 0 \qquad\qquad \text{on } \quad & \partial\Omega
\end{aligned} \tag{2.A.6}$$

if and only if $\sigma_1 = 0$ is a principal eigenvalue for

$$\begin{aligned}
\nabla \cdot d(x)\nabla\psi + \lambda_1^+ m(x)\psi = \sigma\psi \text{ in } \quad & \Omega, \\
\frac{\partial\psi}{\partial\vec{n}} = 0 \qquad\qquad \text{on } \quad & \partial\Omega.
\end{aligned} \tag{2.A.7}$$

If $\int_\Omega m(x)dx > 0$ then by substituting $\psi = 1$ into the formula (2.11a) we see that $\sigma_1 \ge \int_\Omega m dx/|\Omega| > 0$. (Recall that $\beta(x) = 0$ in the case of Neumann conditions.) Suppose

$\int_\Omega m(x)dx = 0$. If $\sigma_1 = 0$ in (2.A.7) let $\psi_1 > 0$ be the associated eigenfunction, multiply (2.A.7) by $1/\psi_1$ and integrate by the divergence theorem to obtain

$$-\int_\Omega \nabla\left(\frac{1}{\psi_1}\right) \cdot d\nabla\psi_1 = 0. \tag{2.A.8}$$

(Here we have used the assumption that $\int_\Omega m(x)dx = 0$.) Since $\nabla(1/\psi_1) = -\nabla\psi_1/\psi_1^2$, we have

$$\nabla(1/\psi_1) \cdot \nabla\psi_1 = -|\nabla_1\psi_1|^2/\psi_1^2 = -|\nabla(\ln\psi_1)|^2$$

so that (2.A.8) implies

$$\int_\Omega d|\nabla(\ln\psi_1)|^2 dx = 0$$

so $\nabla \ln\psi_1 = 0$ and ψ_1 is constant. We then have from (2.A.7) that $m(x) = 0$ on Ω, but we are assuming that $m(x) > 0$ on an open subset of Ω, so that is impossible. Thus, we must have $\int_\Omega mdx < 0$ if (2.A.6) is to have a positive principal eigenvalue.

Suppose that $\int_\Omega mdx < 0$. Let

$$\mu_1 = \sup_{\substack{\phi\in W^{1,2}(\Omega)\\ |\nabla\phi|\neq 0}} \left\{ \frac{\int_\Omega m\phi^2 dx}{\int_\Omega d|\nabla\phi|^2 dx} \right\}.$$

Since $m(x)$ is positive on an open set we have $\mu_1 > 0$, and we may choose a sequence $\{\phi_n\}$ such that

$$0 < \frac{\int_\Omega m\phi_n^2}{\int_\Omega d|\nabla\phi_n|^2 dx} \to \mu_1 \quad \text{as} \quad n \to \infty.$$

We may normalize the sequence by requiring $\int_\Omega \phi_n^2 dx = 1$. Since $\int_\Omega mdx < 0$, Lemma 2.A.3 implies that $\int_\Omega d|\nabla\phi_n|^2 dx \geq c_0 > 0$ for some $c_0 > 0$, so that $\mu_1 \leq ||m||_\infty/c_0 < \infty$. By passing to a subsequence if necessary we may assume that

$$\frac{\int_\Omega m\phi_n^2 dx}{\int_\Omega d|\nabla\phi_n|^2 dx} \geq \mu_1/(1 + 1/n),$$

so that

$$-\mu_1\int_\Omega d|\nabla\phi_n|^2 dx + \int_\Omega m\phi_n^2 dx \geq (-1/n)\int_\Omega m\phi_n^2 dx \geq (-1/n)||m||_\infty.$$

It follows that by taking $\lambda_1^+ = 1/\mu_1$ we get

$$\sigma_1 = \sup\left\{(1/\mu_1)\left[-\mu_1\int_\Omega d|\nabla\phi|^2dx + \int_\Omega m\phi^2dx\right] : \int_\Omega \phi^2 = 1\right\} \geq 0.$$

Also, $\mu_1\int_\Omega d|\nabla\phi|^2 \geq \int_\Omega m\phi^2dx$ for any ϕ, so $-\mu_1\int_\Omega d|\nabla\phi|^2dx + \int_\Omega m\phi^2dx \leq 0$, for any ϕ, so that $\sigma_1 \leq 0$. It follows that for $\lambda_1^+ = 1/\mu_1$, the principal eigenvalue σ_1 in (2.A.7) is zero, so that $\lambda_1^+ > 0$ is indeed a principal eigenvalue for (2.A.6).

Proof of Theorem 2.7: We consider first the case where $\beta(x) > 0$ on an open subset of $\partial\Omega$ or where Dirichlet boundary conditions are imposed on all or part of $\partial\Omega$. Let $\{m_n(x)\}$ be a sequence of bounded measurable functions with $||m_n||_\infty < M_1$ and such that (2.32) holds for $\psi \in L^1(\Omega)$ provided $\psi \geq 0$ almost everywhere. Suppose that contrary to the assertion of Theorem 2.7 we have $\lambda_1^+(m_n) \not\to \infty$ as $n \to \infty$. By passing to a subsequence we may assume that $\lambda_1^+(m_n) \leq \Lambda_0$ for some $\Lambda_0 < \infty$. Let ϕ_n be the eigenfunction corresponding to $\lambda_1^+(m_n)$, normalized by $\int_\Omega d|\nabla\phi_n|^2dx + \int_{\partial\Omega}\beta\phi_n^2dS = 1$ (or, in the case of Dirichlet boundary conditions, $\int_\Omega d|\nabla\phi_n|^2dx = 1$.) By Lemma 2.A.2, or in the case of Dirichlet boundary conditions, by Poincaré's Inequality, the sequence $\{\phi_n\}$ is uniformly bounded in $W^{1,2}(\Omega)$ (or $W_0^{1,2}(\Omega)$ in the Dirichlet case) so that by the compact embedding of $W^{1,2}(\Omega)$ into $L^2(\Omega)$ there is a subsequence of $\{\phi_n\}$ which converges in $L^2(\Omega)$. Passing to the subsequence we may assume $\phi_n \to \phi$ in $L^2(\Omega)$. By multiplying (2.13) by ϕ_n and by using the definition of a $W^{1,2}$ solution to (2.13) (which means, in effect, multiplying (2.13) by ϕ_n and using Green's formula) we have

$$1 = \int_\Omega d|\nabla\phi_n|^2dx + \int_{\partial\Omega}\beta\phi_n^2dS = \lambda_1^+(m_n)\int_\Omega m_n\phi_n^2dx \qquad \text{(2.A.9)}$$

(omit the integral involving β in the Dirichlet case). Recall that we supposed $\lambda_1^+(m_n) \leq \Lambda_0$ for some Λ_0. By (2.A.9) we have

$$1 = \lambda_1^+(m_n)\left[\int_\Omega m_n(\phi_n^2 - \phi^2)dx + \int_\Omega m_n\phi^2dx\right]. \qquad \text{(2.A.10)}$$

Since $\phi_n \to \phi$ in $L^2(\Omega)$, the first integral on the right in (2.A.10) goes to zero as $n \to \infty$. By hypothesis (2.32), we may assume the second integral also goes to zero as $n \to \infty$. Letting $n \to \infty$ in (2.A.10) thus yields the contradiction $1 = 0$, so the sequence $\{\lambda_1^+(m_n)\}$ cannot have a bounded subsequence and we must therefore have $\lim_{n\to\infty}\lambda_1^+(m_n) = \infty$.

In the case of Neumann boundary conditions, we must proceed slightly differently. Again, suppose that $\lambda_1^+(m_n) \not\to \infty$ as $n \to \infty$, so that by passing to a subsequence we may assume $\lambda_1^+(m_n) \leq \Lambda_0$. Normalize the sequence of eigenfunctions so that $\int_\Omega \phi_n^2dx = 1$. We must then have $\int_\Omega d|\nabla\phi_n|^2dx \geq c_0 > 0$. If not we may proceed as in the proof of Lemma 2.A.3 and by passing to a subsequence obtain a sequence $\{\phi_n\}$ with $\int d|\nabla\phi_n|^2dx \leq 1/n$. As in the proof of Lemma 2.A.3 that sequence must have a subsequence converging in $W^{1,2}(\Omega)$.

As noted, σ_1 measures the average growth rate over Ω in the model (7.9). It will be key to a prediction of persistence (0 unstable) versus extinction (0 stable). The eigenvalue λ_1^+ captures essential geometric features of the focal habitat patch under the assumptions of (7.9).

Theorem 7.6 (CC Thm 2.6) Suppose that λ is a positive parameter and that either

$\beta(x) > 0$ on an open subset of $\partial\Omega$

or

the boundary condition is Dirichlet on part of $\partial\Omega$

or

$$\int_\Omega m(x)\,dx < 0.$$

The principal eigenvalue σ_1 of

$$\nabla \cdot d(x) \nabla \psi + \lambda m(x) \psi = \sigma \psi \quad \text{in } \Omega$$

$$d(x) \frac{\partial \psi}{\partial \eta} + \beta(x) \psi = 0 \quad \text{on } \partial\Omega$$

is positive $\iff 0 < \lambda_1^+ < \lambda$, where

λ_1^+ is given by (7.12a), (7.12b) or (7.14).

Proof: Let us suppose $\beta(x) > 0$ on an open subset of $\partial\Omega$. (The other cases will be Exercise 4 of Option B, assuming we get to this point in time.)

Let ψ_1 be the eigenfunction for σ_1, so that

$$\nabla \cdot d(x) \nabla \psi_1 + \lambda m(x) \psi_1 = \sigma_1 \psi_1$$

Multiplying by ψ_1 and integrating over Ω yields

$$\int_\Omega \psi_1 \nabla \cdot d(x)\nabla\psi_1 + \lambda\int_\Omega m(x)\psi_1^2 = \sigma_1\int_\Omega \psi_1^2 .$$

Now $\operatorname{div}(\psi_1(d(x)\nabla\psi_1))$

$$= d(x)|\nabla\psi_1|^2 + \psi_1\nabla\cdot(d(x)\nabla\psi_1)$$

so that

$$\int_\Omega \psi_1 \nabla\cdot(d(x)\nabla\psi_1)$$

$$= \int_\Omega \operatorname{div}(\psi_1 d(x)\nabla\psi_1) - \int_\Omega d(x)|\nabla\psi_1|^2$$

$$= \int_{\partial\Omega} \psi_1 d(x)\nabla\psi_1 - \int_\Omega d(x)|\nabla\psi_1|^2$$

$$= -\int_{\partial\Omega}\beta(x)\psi_1^2 - \int_\Omega d(x)|\nabla\psi_1|^2,$$ so that

$$-\int_\Omega d|\nabla\psi_1|^2 dx - \int_{\partial\Omega}\beta\psi_1^2 dS + \lambda\int_\Omega m\psi_1^2 dx = \sigma_1\int_\Omega \psi_1^2 dx$$

Now $$\frac{1}{\lambda_1^+} = \max_{\substack{\phi\in W^{1,2}(\Omega)\\ \phi\neq 0}} \left[\frac{\int_\Omega m\phi^2 dx}{\int_\Omega d|\nabla\phi|^2 + \int_{\partial\Omega}\beta\phi^2}\right]$$

$$\Rightarrow \int_\Omega m\psi_1^2 \leq \frac{1}{\lambda_1^+}\left[\int_\Omega d|\nabla\psi_1|^2 dx + \int_{\partial\Omega}\beta\psi_1^2 dS\right]$$

$$\Rightarrow \quad \sigma_1 \int_\Omega \psi_1^2 dx$$

$$\leq \left(\frac{\lambda}{\lambda_1^+} - 1\right)\left[\int_\Omega d|\nabla\psi_1|^2 + \int_{\partial\Omega}\beta\psi_1^2\right]$$

So $\lambda < \lambda_1^+ \Rightarrow \sigma_1 < 0$

and

$$\sigma_1 > 0 \Rightarrow \lambda_1^+ < \lambda .$$

Suppose now that ϕ_1 is the eigenfunction for λ_1^+, so that

$$\nabla \cdot d(x)\nabla\phi_1 + \lambda_1^+ m(x)\phi_1 = 0$$

in Ω. Multiply by ϕ_1 and integrate.

$$\int_\Omega \phi_1 \nabla\cdot d(x)\nabla\phi_1 + \lambda_1^+\int_\Omega m(x)\phi_1^2 = 0 .$$

As in the preceding calculation,

$$\int_\Omega \phi_1 \nabla\cdot(d(x)\nabla\phi_1) = -\int_\Omega d(x)|\nabla\phi_1|^2 - \int_{\partial\Omega}\beta(x)\phi_1^2$$

$\Rightarrow$

$$-\int_\Omega d|\nabla\phi_1|^2 - \int_{\partial\Omega}\beta\phi_1^2\,dS = -\lambda_1^+\int_\Omega m\phi_1^2\,dx$$

Since ϕ_1 is an appropriate test function

$$\sigma_1 \geq \frac{\left(-\int_\Omega d|\nabla\phi_1|^2 + \lambda\int_\Omega m\phi_1^2\,dx - \int_{\partial\Omega}\beta\phi_1^2\,dS\right)}{\int_\Omega \phi_1^2}$$

$$= \frac{(\lambda-\lambda_1^+)\int_\Omega m\phi_1^2\,dx}{\int_\Omega \phi_1^2\,dx}$$

By (7.12a), $\lambda_1^+ > 0 \Rightarrow \int_\Omega m\phi_1^2\,dx > 0$.

So we get

$$\lambda > \lambda_1^+ \Rightarrow \sigma_1 > 0$$

$$\sigma_1 < 0 \Rightarrow \lambda < \lambda_1^+$$

So we get $\sigma_1 > 0 \iff \lambda_1^+ < \lambda$.

Notes: (i) As long as $\beta > 0$ or Dirichlet conditions are imposed somewhere along the boundary, a large diffusion coefficient can lead to a rapid loss of individuals across $\partial\Omega$ and a prediction of extinction ($\sigma_1 < 0$); witness

$$\nabla \cdot \left(\frac{d(x)}{\lambda} \nabla \psi \right) + m(x)\psi = \frac{\sigma_1}{\lambda} \psi$$

for $\lambda < \lambda_1^+$.

(ii) If $\beta \equiv 0$ (Neumann case), the only mechanism that might cause loss of population is dispersal into regions where the local population growth rate is negative. (7.13) means the average per capita growth rate is negative. If individuals disperse rapidly they effectively average the local growth rate. So $\sigma_1 < 0$ if (7.13) holds and $\lambda < \lambda_1^+$.

Habitat fragmentation

Effects of fragmentation

$\left.\begin{array}{l}\text{Lovejoy et al (1986)}\\ \text{McKelvey et al (1986)}\\ \text{Quinn and Karr (1986)}\\ \text{Fagan et al (1999)}\end{array}\right\}$ some ecological references.

Consider a situation in which the local population growth rate $m(x)$ is positive in favorable regions but negative in unfavorable ones and examine how the arrangement of those regions affects the eigenvalues λ_1^+ and σ_1 in models such as (7.9).

It is reasonable to expect that an environment where favorable and unfavorable regions are closely intermingled will be less suitable than one where there are large regions of favorable habitat, even if the total amounts of favorable and unfavorable habitat are the same in the two environments. Such is the case since the chances of an individual dispersing into an unfavorable region are greater if the favorable regions are small and close to unfavorable ones.

However, rearranging the location of favorable regions will typically increase $m(x)$ in some places and decrease it in others, So we can not make a simple comparison on the basis of monotonicity.

Note that for (7.9), σ_1 in (7.10) is positive

$\Leftrightarrow$ $\lambda_1^+ < 1$, where $\lambda_1^+ = \lambda_1^+(m)$ is as in

(7.12a), (7.12b) or (7.14).

Theorem 7.7 (CC Thm 2.7) (Cantrell and Cosner 1989)

Suppose that the domain Ω and functions $d(x)$, $\beta(x)$ and

$m_n(x)$, $n = 1, 2, \cdots$, satisfy the hypotheses of

Theorem 7.1, and that $\beta(x) > 0$ on a subset of $\partial\Omega$

or that the boundary condition $d(x)\frac{\partial u}{\partial \eta} + \beta(x)u = 0$

is replaced with the condition $u = 0$ on all or part of $\partial\Omega$.

Suppose further that there is a constant $M_1 > 0$ so

that

$$\|m_n\|_\infty \le M_1$$

for all n. Denote the positive eigenvalue in (7.11) with

$m = m_n(x)$ by $\lambda_1^+(m_n)$. Then

$$\lambda_1^+(m_n) \to \infty \iff \limsup_{n\to\infty} \int_\Omega m_n \psi \, dx \le 0$$

for all $\psi \in L^1(\Omega)$ with $\psi \geq 0$ a.e.

If $\beta \equiv 0$, the conclusion remains valid under the additional hypothesis that there is an $M_0 > 0$ so that

$$\int_\Omega m_n \, dx \le -M_0$$

for $n = 1, 2, \ldots$.

Note: A proof is included in the notes.

It follows that by taking $\lambda_1^+ = 1/\mu_1$ we get

$$\sigma_1 = \sup\left\{(1/\mu_1)\left[-\mu_1\int_\Omega d|\nabla\phi|^2 dx + \int_\Omega m\phi^2 dx\right] : \int_\Omega \phi^2 = 1\right\} \geq 0.$$

Also, $\mu_1 \int_\Omega d|\nabla\phi|^2 \geq \int_\Omega m\phi^2 dx$ for any ϕ, so $-\mu_1 \int_\Omega d|\nabla\phi|^2 dx + \int_\Omega m\phi^2 dx \leq 0$, for any ϕ, so that $\sigma_1 \leq 0$. It follows that for $\lambda_1^+ = 1/\mu_1$, the principal eigenvalue σ_1 in (2.A.7) is zero, so that $\lambda_1^+ > 0$ is indeed a principal eigenvalue for (2.A.6).

Proof of Theorem 2.7: We consider first the case where $\beta(x) > 0$ on an open subset of $\partial\Omega$ or where Dirichlet boundary conditions are imposed on all or part of $\partial\Omega$. Let $\{m_n(x)\}$ be a sequence of bounded measurable functions with $||m_n||_\infty < M_1$ and such that (2.32) holds for $\psi \in L^1(\Omega)$ provided $\psi \geq 0$ almost everywhere. Suppose that contrary to the assertion of Theorem 2.7 we have $\lambda_1^+(m_n) \not\to \infty$ as $n \to \infty$. By passing to a subsequence we may assume that $\lambda_1^+(m_n) \leq \Lambda_0$ for some $\Lambda_0 < \infty$. Let ϕ_n be the eigenfunction corresponding to $\lambda_1^+(m_n)$, normalized by $\int_\Omega d|\nabla\phi_n|^2 dx + \int_{\partial\Omega} \beta\phi_n^2 dS = 1$ (or, in the case of Dirichlet boundary conditions, $\int_\Omega d|\nabla\phi_n|^2 dx = 1$.) By Lemma 2.A.2, or in the case of Dirichlet boundary conditions, by Poincaré's Inequality, the sequence $\{\phi_n\}$ is uniformly bounded in $W^{1,2}(\Omega)$ (or $W_0^{1,2}(\Omega)$ in the Dirichlet case) so that by the compact embedding of $W^{1,2}(\Omega)$ into $L^2(\Omega)$ there is a subsequence of $\{\phi_n\}$ which converges in $L^2(\Omega)$. Passing to the subsequence we may assume $\phi_n \to \phi$ in $L^2(\Omega)$. By multiplying (2.13) by ϕ_n and by using the definition of a $W^{1,2}$ solution to (2.13) (which means, in effect, multiplying (2.13) by ϕ_n and using Green's formula) we have

$$1 = \int_\Omega d|\nabla\phi_n|^2 dx + \int_{\partial\Omega} \beta\phi_n^2 dS = \lambda_1^+(m_n)\int_\Omega m_n\phi_n^2 dx \tag{2.A.9}$$

(omit the integral involving β in the Dirichlet case). Recall that we supposed $\lambda_1^+(m_n) \leq \Lambda_0$ for some Λ_0. By (2.A.9) we have

$$1 = \lambda_1^+(m_n)\left[\int_\Omega m_n(\phi_n^2 - \phi^2)dx + \int_\Omega m_n\phi^2 dx\right]. \tag{2.A.10}$$

Since $\phi_n \to \phi$ in $L^2(\Omega)$, the first integral on the right in (2.A.10) goes to zero as $n \to \infty$. By hypothesis (2.32), we may assume the second integral also goes to zero as $n \to \infty$. Letting $n \to \infty$ in (2.A.10) thus yields the contradiction $1 = 0$, so the sequence $\{\lambda_1^+(m_n)\}$ cannot have a bounded subsequence and we must therefore have $\lim_{n\to\infty} \lambda_1^+(m_n) = \infty$.

In the case of Neumann boundary conditions, we must proceed slightly differently. Again, suppose that $\lambda_1^+(m_n) \not\to \infty$ as $n \to \infty$, so that by passing to a subsequence we may assume $\lambda_1^+(m_n) \leq \Lambda_0$. Normalize the sequence of eigenfunctions so that $\int_\Omega \phi_n^2 dx = 1$. We must then have $\int_\Omega d|\nabla\phi_n|^2 dx \geq c_0 > 0$. If not we may proceed as in the proof of Lemma 2.A.3 and by passing to a subsequence obtain a sequence $\{\phi_n\}$ with $\int d|\nabla\phi_n|^2 dx \leq 1/n$. As in the proof of Lemma 2.A.3 that sequence must have a subsequence converging in $W^{1,2}(\Omega)$.

If the subsequence converges to ϕ, we must have $\int_\Omega d|\nabla\phi|^2 dx = 0$ but $\int_\Omega \phi^2 dx = 1$ so $\phi = 1/|\Omega|^{1/2}$. We then have

$$\begin{aligned} 0 \le 1/\lambda_1^+(m_n)\int_\Omega d|\nabla\phi_n|^2 dx &= \int_\Omega m_n\phi_n^2 dx \\ &= \left[\int_\Omega m_n(\phi_n^2 - \phi^2)dx + \int_\Omega m_n\phi^2\right]. \end{aligned}$$

By hypothesis the second integral on the right is bounded by $-M_0/|\Omega| < 0$ for all n. The first integral on the right goes to zero as $n \to \infty$. Thus, by letting $n \to \infty$ we obtain $0 \le -M_0/|\Omega|$, a contradiction. Thus, we must have $\int d|\nabla\phi_n|^2 dx \ge c_0 > 0$ for some c_0. We can renormalize the sequence $\{\phi_n\}$ so that $\int_\Omega d|\nabla\phi_n|^2 dx = 1$ and $\int_\Omega \phi_n^2 dx \le 1/c_0$ by dividing each element ϕ_n by $\left(\int_\Omega d|\nabla\phi_n|^2 dx\right)^{1/2}$, so that the sequence is uniformly bounded in $W^{1,2}(\Omega)$. From this point on we may proceed as in the case of $\beta > 0$ on part of $\partial\Omega$ or the Dirichlet case.

Conversely, let us assume that $\lambda_1^+(m_n) \to \infty$ as $n \to \infty$. If there exists $\psi \in L^1(\Omega)$ with $\psi \ge 0$ and $\limsup_{n\to\infty}\int_\Omega m_n\psi dx = \epsilon > 0$ then we have $\int_\Omega m_n\psi dx \ge \epsilon/2$ for some subsequence of $\{m_n\}$. We can approximate $\sqrt{\psi}$ as closely as we wish in $L^2(\Omega)$ with a smooth function ϕ_0, so that $\int_\Omega m_n\phi_0^2 dx \ge \epsilon/4$. We have

$$\begin{aligned} \frac{1}{\lambda_1^+(m_n)} &= \sup_{\substack{\phi\in W^{1,2}(\Omega)\\ \phi\ne 0}}\left(\frac{\int_\Omega m_n\phi^2 dx}{\int_\Omega d|\nabla\phi|^2 dx + \int_{\partial\Omega}\beta\phi^2 dS}\right) \\ &\ge \frac{\int_\Omega m_n\phi_0^2 dx}{\int_\Omega d|\nabla\phi_0|^2 dx + \int_{\partial\Omega}\beta\phi_0^2 dS} \\ &\ge \frac{\epsilon/4}{\int_\Omega d|\nabla\phi_0|^2 dx + \int_{\partial\Omega}\beta\phi_0^2 dS}. \end{aligned} \tag{2.A.11}$$

Inequality (2.A.11) implies that $1/\lambda_1^+(m_n)$ is bounded below by a positive quantity independent of n, so we cannot have $\lambda_1^+(m_n) \to \infty$. Thus, if $\lambda_1^+(m_n) \to \infty$ as $n \to \infty$ we must have $\limsup_{n\to\infty}\int_\Omega m_n\psi dx \le 0$ for any $\psi \in L^2(\Omega)$ with $\psi \ge 0$ almost everywhere. This part of the proof is essentially the same in all cases. In the Neumann case, if $\int_\Omega m_n\psi dx \ge 0$ then ψ and hence ϕ_0 must be nonconstant since $\int_\Omega m_n dx \le -M_0$, so we will have $\int_\Omega d|\nabla\phi_0|^2 dx > 0$.

Corollary 7.8 (CC Corollary 2.8). If $\limsup_{n\to\infty} \int_\Omega m_n \psi \, dx \le 0$ for all $\psi \in L^1(\Omega)$ with $\psi \ge 0$ a.e., then (7.9) predicts extinction for n sufficiently large.

Proof: By Theorem 7.6, $\sigma_1 > 0 \iff \lambda_1^+(m) < 1$. Since $\lambda_1^+(m_n) \to \infty$ as $n \to \infty$, $\sigma_1 < 0$ for n sufficiently large.

Notes: (i) $\limsup_{n\to\infty} \int_\Omega m_n \psi \, dx \le 0$ for $\psi \in L^1(\Omega)$ with $\psi \ge 0$ a.e. $\Rightarrow$ for any suitable ψ on Ω, the sequence of numbers $\{\int_\Omega m_n \psi \, dx\}$ has the property that given $\varepsilon > 0$, $\int_\Omega m_n \psi \, dx < \varepsilon$ for all sufficiently large n. Such a condition clearly holds if $m_n(x)$ is small on all or most of Ω, but it can also hold if $m_n(x)$ is large on parts of Ω but negative on other parts, provided the regions where $m_n > 0$ and $m_n < 0$ become smaller and closer together as $n \to \infty$.

(ii) Let $\Omega = (0, \ell)$ and $m_n(x) = \sin(nx)$

Riemann-Lebesgue Lemma (Royden 1968 or Rudin 1966)

$$\Rightarrow \quad \limsup_{n \to \infty} \int_0^\ell \sin(nx) \psi \, dx \leq 0$$

So the model

$$u_t = d u_{xx} + \sin(nx) u \quad \text{in } (0, \ell) \times (0, \infty)$$

$$u = 0 \quad \text{on } \partial(0, \ell) \times (0, \infty)$$

predicts extinction if n is sufficiently large.

The functions $\sin(nx)$ have maximum value 1 and will be positive on approximately half of $(0, \ell)$ for large n. However, as $n \to \infty$ the regions where $\sin(nx) > 0$ become smaller and they alternate with regions where $\sin(nx) < 0$.

Proposition 7.9 (CC Proposition 2.9). Suppose that σ_1 is the principal eigenvalue of (7.10). Suppose that there is a subdomain $\Omega' \subseteq \Omega$ so that the principal eigenvalue σ_1' for

$$\nabla \cdot d(x) \nabla \psi + m(x) \psi = \sigma \psi \quad \text{in } \Omega'$$
$$\psi = 0 \quad \text{on } \partial\Omega'$$

is positive. Then $\sigma_1 > 0$, so (7.9) predicts persistence.

Proof: Let ψ_1' be a eigenfunction corresponding to σ'. Extend ψ' to be identically zero outside Ω'. The resulting function, call it ψ_1 is then in $W^{1,2}(\Omega)$. So it can be used as a test function in Theorem 7.1.

Thus the maximum over all test

functions must exceed this value, which is σ_1'.

So $\sigma \geq \sigma_1' > 0$.